Economic Modelling xxx (xxxx) xxx

Contents lists available at ScienceDirect

Economic Modelling

journal homepage: www.journals.elsevier.com/economic-modelling

Expectile CAPM[☆]

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ARTICLE INFO

JEL classification:
G11
G12

Keywords:
CAPM
Expectile
Sentiment
Portfolio optimization
Pricing kernel
Momentum

ABSTRACT

Conventional wisdom suggests that the uncertainty of uninformed noise-traders' sentiment deters rational traders' arbitrage activities. However, nowadays, social media have made the public sentiment highly predictable, whereas the CAPM-motivated beta-return relation still does not hold in practice. This study advances an argument that the sentiment can also be brought about by rational, sophisticated investors' use of psychological insight; resultantly, the arbitrage activities are demotivated by their own sentiment, rather than deterred by noise-traders' sentiment risk. The proposed expectile CAPM provides a parsimonious way to account for this claim, and leads to a sentiment-based functional form of pricing kernel.

1. Introduction

Noise traders are uninformed, irrational and have erroneous stochastic beliefs. When rational arbitrageurs have large exposure to the uncertainty of noise trader's sentiment (also known as uninformed noise traders' sentiment risk or noise traders' risk), their arbitrage activities can be deterred. Therefore, noise traders' sentiment will be priced as a risk factor. However, nowadays, social media, such as Facebook, Twitter and Pinterestare, are representing public sentiment. Can noise traders learn?

When a message of interest is posted on social networks or microblogging platforms, its informational content, together with the associated opinions of its readers, can rapidly spread. Information of valuable financial interest, thus, hits the markets, giving the opportunity for uninformed traders to investigate and adopt market sentiment. Although social media cannot remove the randomness associated with noise traders' beliefs completely, market sentiment has been highly predictable. The existing approaches to extracting sentiment from social media can be, but are not limited to, lexicon based, machine learning based, or their combination. Furthermore, the evolution of a market structure constructed from stock market prices is often observed to reconcile with Twitter sentiment signals. Why is it, then, that the CAPM-motivated positive relation between beta and returns still does not hold in practice?

This study provides new perspectives: sentiment can also be brought about by rational, sophisticated investors' use of psychological insight; then the arbitrage activities can be demotivated by their own sentiment, rather than deterred by noise-traders' sentiment risk. Thus, the predictability of the market sentiment does not prevent the sentiment from being a pricing factor. What's more, models based on noise-traders' sentiment risk suggest that policy makers prompt educating "naïve" investors about their behavioral biases and thus preclude them from trading on meaningless signals. Based on the theory developed in this paper, however, this action might not enhance the efficiency of equity market prices effectively, because sentiment is modeled as brought about by rational, sophisticated investors.

This study sheds light on this argument by proposing a sentiment-based extension of Merton (1969, 1971, 1973)'s optimality and equilibrium theories. In particular, we define sentiment portfolio optimization and show its equivalence with Merton (1971)'s portfolio optimization under the subjective (distorted) measure. This equivalence leads to the expectile CAPM and a new functional form of pricing kernel. In their seminal work, Fama and French (1992) documented that the CAPM-motivated positive relation between beta and returns does not hold in practice. They consider and overrule two explanations for this

☆ We thank Rong Chen, Robert Durand, William Greene, Bing Han, and Chu Zhang for helpful discussions and thank participants of European Financial Management Association (EFMA) Annual Meeting and Asian Finance Association (AsiaFA) Annual Meeting. We acknowledge the financial support from the National Natural Science Foundation of China (grant number: 71871190, 71790601, 71471155).

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https://doi.org/10.1016/j.econmod.2019.09.049
Received 31 March 2019; Received in revised form 12 August 2019; Accepted 29 September 2019
Available online xxxx

Please cite this article as: Hu, W., Zheng, Z., Expectile CAPM, Economic Modelling, https://doi.org/10.1016/j.econmod.2019.09.049

finding: 1) that beta is measured noisily, and 2) other variables that do explain returns are correlated with "true" betas. The expectile CAPM provides a theoretical basis for the hypotheses, by claiming that sentiment affects both the quantity of risk and the market price of risk.

This two-fold effect is of immense importance, as excessive pessimism/optimism and ambiguity aversion/overconfidence form two different layers of sentiment. On the one hand, models of ambiguity aversion based on multiple-priors preferences (Chen and Epstein, 2002) can be interpreted in terms of pessimism, but ambiguity can enter into the market price of risks (uncertainty) only. On the other hand, in models of overconfidence, i.e., the underestimation of volatility, only the quantity of risks can be affected. However, empirical estimates of optimism and overconfidence tend to comove over time. The expectile-based framework established in this study provides a parsimonious way to account for this finding.

This study also bridges a gap in the sense that most existing empirical studies focus on the ability of sentiment to explain the time-series of the cross-section (as a contrarian predictor), rather than its role as a pricing factor. However, the proposed expectile CAPM introduces a linkage between systematic (idiosyncratic) risk and comonotonic (non-comonotonic) risk,[1] which are equivalent to each other when market sentiment is absent. This linkage leads to a novel risk channel reproducing the cross-sectional momentum. Specifically, the non-comonotonic risk could constitute a large proportion of the systematic risk. While the winner portfolio has a low comonotonic risk, it has a high non-comonotonic risk. The latter could numerically overweigh the former in determining the systematic risk. It follows that the rankings of the systematic risk inherent in stocks could be substantially altered when market sentiment is incorporated. That explains Yu and Yuan (2011)'s empirical findings that there is a strong positive return-risk trade-off when sentiment is low, but little, if any, relationship when sentiment is high.

The concept of rational, sophisticated investors' sentiment is motivated by two psychological observations. The first one is related to univariate decision making. "Is the glass half empty or half full?" indicates rhetorically that pessimism or optimism is a disposition, and it could be unrelated to inexperience, naivety, ignorance, or poor understanding. It is formed from deep perception cues, such as "thinking the worst, leads to the best". Nofsinger (2017) suggests this could be affected by, e.g., genetics, emotion, pride, regret, social interaction or familiarity. However, getting to the bottom of the "biology of investment" is beyond the scope of this research.

Another psychological observation, stated as an aphorism, indicates the importance of sentiment in framing multivariate decisions: "a pessimistic father of two sons, who are an umbrella seller and a fisherman, worries all the time, no matter whether tomorrow is sunny or rainy"; however, a risk-averse father thinks that his family is immune from weather risks. Although drawing little attention in the literature, this psychological disposition sheds light on the critical points distinguishing the sentiment-driven and the rational approaches to decision making: 1) sentiment-driven traders reduce the dimensions of the problem first, maximally, then evaluate the outcome of each low-dimensional sub-problem with their distorted priors, and then aggregate the results; 2) for each sub-problem, traders apply their sentiment to the corresponding marginal prior only and, hence, a negative correlation does not necessarily diversify the risk.

In this paper, we first assume a representative agent model, and then study the aggregation of the heterogeneities of sentiment, CARA risk aversion, and initial wealth. We postulate that the agent asymmetrically weights the probability of the exceedances beyond the expectile and of the rest. Based on that distortion, we define the expectile of the original undistorted prior and the value of the scoring function at the expectile, concurrently, as a reward-risk measure pair for a univariate. We then conceptually extend the expectile into the multivariate with/without information reduction and, to distinguish them, we give the names, *photographic* and *holographic expectile*, respectively, to indicate the intuition that holography takes single holograms from different viewpoints to reduce the dimensions, while photography reproduces a rudimentary image of the full dimension. We employ intuition as a metaphor for sentiment trading.

2. Literature review

The whole literature of sentiment based on noise trading is motivated by Black (1986), and there is investor-level evidence supporting this sentiment risk channel. For instance, Vissing-Jorgensen (2003) shows that inexperienced investors had the most optimistic beliefs during the tech stock boom (a period of general optimism); Puri and Robinson (2007) find that individuals with optimistic beliefs are more likely to invest in individual stocks.

DeLong et al. (1990), Barberies et al. (1998) and Kogan et al. (2006) detected uninformed traders' trading noise, and documented the stylized fact that noise traders' risk can deter rational arbitrageurs' arbitrage activities. In Dumas et al. (2009)'s model, the resultant excess movement of stock return arising from the vagaries of the overconfident population is regarded a "sentiment" factor. Barone-Adesi et al. (2014) distinguished optimism and overconfidence, and documented their comovement. Lee et al. (1991); Pontiff (1996); Kumar and Lee (2006); Baker and Wurgler (2006, 2007); Antoniou et al. (2015) proposed that noise traders' sentiment should be priced as a risk factor. Although the sentiment risk factor partially alleviate the pricing anomalies, Brown and Cliff (2005) and Lemmon and Portniaguina (2006) still observed time-series variation of the pricing error with sentiment. That indicates some predictive information that is not completely captured by sentiment as a pricing factor.

Meanwhile, the advancement in Twitter sentiment analysis, such as Kolchyna et al. (2016a, 2016b), Barbosa and Feng (2010), Kouloumpis et al. (2011), confirmed the predictability of market sentiment. Tetlock (2007), using Wall street Journal as example, studied the role of relatively traditional media in giving content to market sentiment. Thus, those studies challenged noise-trader sentiment risk Hypothesis. Accordingly, we proposed a hypothesis of sophisticated traders' sentiment, using of their psychological insight.

The following empirical studies provided sentiment explanations on why the CAPM does not hold: Barberies et al. (1998), Kogan et al. (2006), Antoniou et al. (2015). They all focused on noise traders' rather than sophisticated traders' sentiment. The other legitimate reasons which are not sentiment-based (hence can be treated as controls), have been explored by Merton (1987), Black (1972), Frazzini and Pedersen (2014), Cohen et al. (2005), Hong and Sraer (2016), Kumar (2009), Bali et al. (2011), Campbell et al. (2018), Baker et al. (2011), Buffa et al. (2014).

The following empirical study focused on the ability of sentiment as a contrarian predictor rather than a pricing factor. Brown and Cliff (2004) find little predictive power using their sentiment measures, and Lemmon and Portniaguina (2006) find stronger evidence of sentiment as a contrarian predictor of small stocks and low institutional ownership stocks. Baker and Wurgler (2006) find robust predictability of the time-series of the cross-section using an index based on six proxies of investors' propensity to invest in stocks. In particular, they observe that sentiment has relatively stronger effects on stocks whose valuations are highly subjective and difficult to arbitrage.

The paper proceeds as follows. Section 3 reviews the existing key concepts of the expectile, and provides a new Definition - *holographic expectile* for multivariate. Section 4 formulates the expectile-based sentiment asset pricing framework. Section 5 discusses applications and policy implementations. Section 6 concludes.

[1] Comonotonic risk refers the perfect positive dependence between the components of a random vector. Non-comonotonic risk refers the orthogonal component to the comonotonic one.

3. Expectile

Newey and Powell (1987) first introduced the expectile. The two equivalent definitions are listed below (also see Schnabel and Eilers, 2009):

Definition 1. If $\mathbb{E}|X| < \infty$ exists, for given $\theta \in [0, 1]$, the expectile $\mathbb{E}^\theta(X)$ is defined as

$$\mathbb{E}^\theta(X) \equiv \underset{q}{\operatorname{argmin}} \mathscr{S}^\theta(X; q), \tag{1}$$

where the scoring function $\mathscr{S}^\theta(X; q)$ is

$$\mathscr{S}^\theta(X; q) \equiv (1-\theta)\int_{-\infty}^{q} (X-q)^2 dF(X) + \theta \int_{q}^{\infty} (X-q)^2 dF(X), \tag{2}$$

and $F(X)$ is the cumulative distribution function of random variable X.

Definition 2. . If $|X| < \infty$, for given $\theta \in [0, 1]$, the expectile $\mathbb{E}^\theta(X)$ is defined as the unique solution of the implicit equation for q:

$$q = \frac{(1-\theta)\int_{-\infty}^{q} XdF(X) + \theta \int_{q}^{\infty} XdF(X)}{(1-\theta)F(q) + \theta[1-F(q)]}. \tag{3}$$

We denote $\mathbb{E}^\theta(X) \equiv \mathbb{E}[\pi_X(\theta)X]$, where $\pi_X(\theta) \equiv \frac{(1-\theta)1_{\{X<\mathbb{E}^\theta(X)\}} + \theta 1_{\{X>\mathbb{E}^\theta(X)\}}}{(1-\theta)F(q)+\theta[1-F(q)]}$.

Intuitively, we interpret the expectile as a weighted average equaling the division point, to the left and right side of which, the agent assigns weights $(1-\theta)$ and θ respectively, and then scales the weighted density into a probability one. We define the asymmetric weighting factor θ as a sentiment measure.

For example, if X represents the return on investment, and the agent is pessimistic, then she overweighs the probability of the bad state [i.e., $(1-\theta) > 50\%$] and underweights the probability of the good state (i.e., $\theta < 50\%$); the agent does the opposite when holding an optimistic view.

We then conceptually extend the expectile into a multivariate. The first approach in this study is to reduce the dimensions of the probability distribution, i.e., for any measureable function $h(\cdot, \cdot)$, we define $Z \equiv h(X_1, X_2, \ldots, X_d)$, then according to Definition 2, $\mathbb{E}^\theta[Z] \equiv \mathbb{E}[\pi_Z(\theta)Z]$. We call it the *photographic expectile* of $h(X_1, X_2, \ldots, X_d)$.

The second approach is to define a *holographic expectile* $\mathbb{E}^\theta[h(X_1, X_2, \ldots, X_d)]$ by constructing an asymmetrically-weighted joint distribution $\mathbb{P}^\theta$. As the name implies, a *photographic expectile* is a two-dimensional representation of $(Z \to \mathbb{E}^\theta[Z])$, which can only reproduce a rudimentary image of $((X_1, X_2, \ldots, X_d) \to \mathbb{E}^\theta[h(X_1, X_2, \ldots, X_d)])$; while the *holographic expectile* displays a full multi-dimensional image and adds greater depth to the perception cues that were presented in the agent's original prior.

Many recent attempts e.g., Herrmann et al. (2018) and Maume-Deschamps et al. (2017) have been made on the extension: however, it remains an open question. Incorporating the dependence in a consistent way, and extending the asymmetric weighting scheme into a multivariate, present difficulties.

Herrmann et al. (2018) defined a Geometric expectile vector q of a random vector X as the minimizers of the expected loss based on Λ_θ, i.e., $\mathbb{E}^{\boldsymbol{\theta}}[X] = \underset{q\in\mathbb{R}^d}{argmin}\mathbb{E}[\Lambda_{\boldsymbol{\theta}}(X-q)]$, where $\Lambda_{\boldsymbol{\theta}} : \mathscr{R}^d \to [0,\infty)$, $\| \ t \| \mapsto \Lambda_{\boldsymbol{\theta}}(\| \ t \|) = \frac{1}{2}t_2(t_2 + 2\boldsymbol{\theta} - 1, t)$, $\boldsymbol{\theta}$ is the asymmetric weighting vector, and $\| \cdot \|_2$ is Euclidean norm. However, the expectile calculated based on the marginal distribution from a distorted d-dimensional joint distribution is not consistent with the resultant expectile based on the one dimensional distorted marginal distribution.

Maume-Deschamps et al. (2017) incorporated the dependence structure of X, and defined the multivariate expectiles as $\mathbb{E}^\theta[X] = \underset{q\in\mathbb{R}^d}{argmin}\mathbb{E}[\theta(X-q)_+^\top\Sigma(X-q)_+ + (1-\theta)(X-q)_-^\top\Sigma(X-q)_-]$, where $(t)_+ = ((t_1)_+, \ldots, (t_d)_+)^\top$, and $(t_i)_+$ represents the positive part of t_i; $(t)_- = ((t_1)_-, \ldots, (t_d)_-)^\top$, and $(t_i)_-$ represents the negative part of t_i; Σ is the covariance matrix. The problem inherent in Herrmann et al. (2018)'s Definition still exists. What's more, Maume-Deschamps et al. (2017)'s definition gives zero weight upon the values of variables that are not consistently positive or negative, e.g., $((t_1)_+, (t_1)_-, (t_1)_+ \ldots, (t_d)_-)^\top$. Positive and negative parts of each random variable segment the space into 2^d subsets. Maume-Deschamps et al. (2017)'s definition considered only two of them, and neglected all the rest.

Then, how to define the multivariate expectile? As our purpose is to study how sentiments affect investors' investment decisions, the extension of the expectile should be able to reflect the "A father of two sons" phenomenon. First, a sentiment-driven agent will reduce the dimension of the problem maximally, evaluate the outcome of each low-dimensional sub-problem with her distorted prior, and then aggregate the results; hence, a negative correlation does not necessarily diversify the risk in sentiment decision making. Second, for each sub-problem, the agent applies her sentiment on the corresponding marginal prior only. The name: *holographic expectile*; shows the intuition that holography takes single holograms from different viewpoints to reduce the dimension, while photography reproduces a rudimentary image of the full dimension. We employ the intuition as a metaphor for sentiment trading, and start by defining the multivariate expectile for an independent random vector X.

Definition 3. The *holographic expectile* of measurable function: $h(X_1, X_2, \ldots, X_d)$, where X_i are independent, is

$$\mathbb{E}^\theta[h(X_1, X_2, \ldots, X_d)] \equiv \mathbb{E}\left[h(X_1, X_2, \ldots, X_d)\prod_{i=1}^{d} \pi_{X_i}(\theta)\right] \tag{4}$$

For convenience, we consider only joint normally distributed random variables henceforth. Such restriction is sensible for two reasons. First, it is quite a common assumption that stock price follows geometric Brownian motion. Second, any nonlinear factor model can be turned into a linear one in discrete time by assuming normal returns (See Cochrane, 2010; page 154). We also postulate that for any sub-problem where the dimension has been reduced, if it is still multi-dimensional with a non-diagonal covariance matrix, the agent will orthogonalize the random variables using the Cholesky decomposition procedure with all possible permutations, apply Definition 3 for each permutation, and then take the average.[2]

In our context, the expectile, as a reward measure, is *holographic additive* by Definition and *photographic subadditive* (*superadditive*), i.e., $\mathbb{E}^\theta(Z) < \mathbb{E}^\theta(X) + \mathbb{E}^\theta(Y)$ for $50\% < \theta < 1$ and $\mathbb{E}^\theta(Z) > \mathbb{E}^\theta(X) + \mathbb{E}^\theta(Y)$ for $0 < \theta < 50\%$, where $Z = X + Y$.

Holographic additivity confirms the classical law of one price, i.e., the same portfolio must give the same price with complete information, see Cochrane (2010). *Photographic subadditivity* and *superadditivity* extend the *Law of one price* with information asymmetry. The economic interpretation is that when market pessimism or optimism prevails, information asymmetry leads to different subjective prices for this portfolio. When trading with a counterparty that has only a prior of portfolio return, the party who has information superiority, i.e., having a prior of the joint distribution of individual security returns, could buy individual assets, repackage them and sell the portfolio for a higher price than it costs to assemble it, if market pessimism prevails. Otherwise, they could buy a portfolio and sell its contents, if market optimistic prevails. The

[2] E.g., without loss of generality, we illustrate the above point by assuming a correlated standard normal vector $(X, Y)^\top$ with correlation ρ, and $h(X, Y) = X + XY$. Then the *holographic expectile* is calculated as: $\mathbb{E}^\theta[h(X, Y)] \equiv \mathbb{E}^\theta[X] + \mathbb{E}^\theta[XY] = \mathbb{E}[\pi_X(\theta)X] + \frac{1}{2}\mathbb{E}[(\pi_X(\theta)\rho X^2 + \pi_X(\theta)\pi_{Z_X}(\theta)sign(\rho)\sqrt{1-\rho^2}Z_X X)] + \frac{1}{2}\mathbb{E}[(\pi_Y(\theta)\rho Y^2 + \pi_Y(\theta)\pi_{Z_Y}(\theta)sign(\rho)\sqrt{1-\rho^2}Z_Y Y)] = \Phi + \rho\Psi + sign(\rho)\sqrt{1-\rho^2}\Phi^2$, where $\Phi \equiv \mathbb{E}[\pi_X(\theta)X]$, and $\Psi \equiv \mathbb{E}[\pi_X(\theta)X^2]$, Z_X and Z_Y are standard normal with $\rho_{Z_X,X} = 0$ and $\rho_{Z_Y,Y} = 0$.

instantaneous profits earned by doing so reward the party, who has information superiority, for releasing information to the market.

Expectile has been widely accepted as a risk measure in describing the tail behavior of financial positions. For the first time in the literature, this research employs it as a reward measure to represent an agent's distorted prior, for the following reasons:

First, as a risk measure, the expectile reflects diversification benefits via its *subadditivity*, but allows diversification on comonotonic risks due to its *commonotonic non-additivity* (see Ziegel, 2016). That contradicts intuition. In our context, as a reward measure, the expectile is defined in such a way that it is *holographic additive*, i.e., it is additive when an agent has a prior of the joint distribution of random variables. Meanwhile the expectile is *photographic subadditive* (*superadditive*) when the optimistic (pessimistic) agent has a prior of the distribution of the summed values only. The *holographic additivity* confirms the classical *Law of One Price* with complete information. When information asymmetry exits, the *subadditivity* and *superadditivity* rewards the party with information superiority for providing additional information to the market, and that admits appealing economic interpretations.

Second, the scoring function at the expectile, as a risk measure, then satisfies *subadditivity* and *commonotonic additivity,* which are the required risk-measure properties. What's more, the expectile satisfies other *Coherence* properties, i.e., *positive homogeneity*, *monotonicity*, and *translation invariance*, which are the axioms that any reward measure used for performance comparison, insurance, management and regulation should satisfy.[3]

Third, the expectile is *elicitable* and *law-invariant*, where *elicitability* enables the verifiability of the estimation through back-testing, as well as the comparability of different estimating-procedures under the same measure; and where *law-invariance*, a desired property for a measure when the set of priors is a singleton, ensures that the measure can be completely determined in terms of a probability distribution.[4]

4. Expectile asset pricing

In this section, we introduce a sentiment portfolio optimization problem and derive an expectile CAPM to incorporate market sentiment.

4.1. Model setting

In this subsection, we assume a representative agent model. In the market, there is a traded bond whose price $S_0(t)$ appreciates at a risk-free rate of interest r, thus, evolving according to the differential equation:

$$dS_0(t) = rS_0(t)dt, \qquad S_0(0) = 1 \tag{5}$$

Uncertainty in the market is driven by a d-dimensional standard Brownian motion $W(t) = (W_1(t), \ \ldots, \ W_d(t))^\top$ in $\Re^d$, defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a $d \times d$ correlation matrix $\rho \equiv (\rho_{ij})$, and we denote by $\{\mathscr{F}_t\}$ the $\mathbb{P}$-augmentation of the natural filtration $\mathscr{F}_t^W = \sigma(W(s); 0 \le s \le t)$, with time span $[0, T]$ for some finite $T > 0$.

Primary asset prices $S_i(t), \quad i = 1, \ldots, \quad d$ follow the dynamics of

$$dS_i(t) = S_i(t)[b_i dt + \sigma_i dW_i(t)], \qquad S_i(0) = s_i, \quad i = 1, 2, \ldots, d \tag{6}$$

Here $\sigma \triangleq (\sigma_i)$ is a $d \times 1$ volatility vector, and $b \triangleq (b_i)$, is a $d\times$ 1 drift rate vector. We assume that: r, b, σ and ρ are all constant.

Let $x_t \in (0, \infty)$ be the initial agent's wealth at $t = 0$, or the realized agent's wealth at $t > 0$. The total wealth $X(\cdot) \equiv X^{x_t, \ \varpi}(\cdot)$ follows the dynamic,

$$\begin{aligned} dX(s) = & \sum_{i=1}^{d} \varpi^i(s) X(s) \{b_i ds + \sigma_i dW_i(s)\} \\ & + \left\{1 - \sum_{i=1}^{d} \varpi^i(s)\right\} X(s) r ds - c(s) ds, \forall s \ge t, \text{ and } X(t) \equiv x_t, \end{aligned} \tag{7}$$

where $(\varpi, \ c)$ represents the portfolio/consumption process pair.

Let $U_i : (0, \infty) \to \Re, i = 1, 2$, be strictly increasing, strictly concave, utility functions of class C^1, satisfying,

$$dU_i^{'}(0+) \triangleq \lim_{x \downarrow 0} U_i^{'}(x) = \infty, \quad U_i^{'}(\infty) \triangleq \lim_{x \to \infty} U_i^{'}(x) = 0 \tag{8}$$

Then the *sentiment portfolio optimization*, with initial state $X(t) \equiv x_t$: $\forall$ $t \in [0, T]$ is

$$\mathbb{J}(x_t, t) \triangleq \underset{(\varpi, c) \in \mathscr{A}_0^{'}(x_t, t, \ T)}{\text{esssup}} \ \mathbb{E}^\theta \left[\int_t^T U_1[c(s), s] ds + U_2(X^{x_t, \ \varpi, c}(T)) | \mathscr{F}_t \right], \tag{9}$$

with boundary condition

$$\mathbb{J}(X(T), T) \triangleq U_2(X(T)), \tag{10}$$

We postulate that even though the agent is sentiment driven, she is rational in maximally utilizing the information she has, hence, sentiment distorts the prior of the joint distribution of asset returns rather than the univariate distribution of portfolio return; also, the agent converts a high-dimensional risk source into a linear combination of univariate and independent bivariate risk sources through deriving an expectile Hamilton–Jacobi–Bellman (HJB) of Eq (9). It is important to know that, the agent's choice of approach and the extent to which she reduces dimension, determines the resultant value of expectile. That implies dimensionality reduction is part of the expectile-based modelling.

This optimization is to obtain the indirect utility $\mathbb{J}(x_t, t)$ by choosing (ϖ, c) over the class $\mathscr{A}_0^{'}(x_t, t, T)$. Following the setting of Cvitanić and Karatzas (1992), it is a set of $(\varpi, c) \in \mathscr{A}_0(x_t, t, \ T)$ that satisfy

$$\mathbb{E}\left[\int_t^T U_1^-[c(s), s] ds + U_2^-(X^{x_t, \ \varpi, c}(T)) | \mathscr{F}_t \right] < \infty, \tag{11}$$

where $\mathscr{A}_0(x_t, t, T)$ is a set of admissible pairs ensuring that, for the initial capital, $x_t \in (0, \ \ \infty)$, $X^{x_t, \quad \varpi, c}(s) \ge 0$, $\forall t \le s \le T$, holds almost surely. We postulate that $\mathbb{J}(x_t, t)$ is in $C^{2,1}(\mathscr{R}^+, [0, T])$.

4.2. Expectile CAPM

In this subsection, we solve *sentiment portfolio optimization* and obtain Proposition 1, which provides insight into many aspects of asset pricing, such as optimal wealth allocation and consumption, portfolio rebalancing frequency and risk decomposition.

Proposition 1. Given market sentiment θ such that the $d \times d$ matrix $V_\theta \equiv (\rho_{ij}\sigma_i\sigma_j + sign(\rho_{ij})\sqrt{1 - (\rho_{ij})^2}\,\sigma_i\sigma_j\Phi^2)$ is positive semi-definite (p.s.d.), the optimal portfolio/consumption process pair (ϖ^*, c^*) solving the sentiment portfolio optimization problem is

$$\varpi^*(t) = -\frac{\mathbb{J}_X(X(t), t)}{X(t)\mathbb{J}_{X,X}(X(t), t)} \varphi^\theta, \tag{12}$$

and

$$c^*(t) = I_1(t, \mathbb{J}_X(X(t), t)), \tag{13}$$

where $\varphi^\theta \equiv V_\theta^{-1}\{b + \Phi\sqrt{n}\,\sigma - r1\}$ is a vector of the market price of risk with rebalancing frequency n, $I_1(t, \cdot)$ is the inverse of the function $U_1^{'}(t, \cdot)$, and $\Phi \equiv \mathbb{E}^\theta[\varepsilon]$, with $\varepsilon \sim \mathscr{N}(0, 1)$.

Proof. *See* Appendix A.

[3] See Emmer et al. (2015) for the Definition formulas of *subadditivity*, *commonotonic additivity*, *positive homogeneity*, *monotonicity*, and *translation invariance*.

[4] See Ziegel (2016) for the Definition formulas of *elicitability* and *law-invariance*.

Theorem 1. (*Expectile CAPM*) Suppose the usual assumptions for CAPM apply, except that the agent is driven by sentiment as well rather than purely by rationality, then the sentiment portfolio optimization implies that for any security $i = 1, 2, \dots, d$,

$$b_i + \Phi\sqrt{n}\sigma_i - r = \beta^\theta \left(b_M(t) + \Phi\sqrt{n}\left[\phi^*(t)\right]^\top \sigma - r\right) \tag{14}$$

where $\phi^*(t) = \frac{\varpi^*(t)}{\sum_{i=1}^d \varpi_i^*(t)}$; $b_M(t) = \sum_{i=1}^d \phi_i^*(t) b_i$; $\sigma_{iM}(t) \equiv \sum_{j=1}^d \rho_{ij}(t)\sigma_i(t)\sigma_j(t)\phi_j^*(t)$ $\sigma_{MM}(t) \equiv \sum_{i=1}^d \phi_i^*(t)\sigma_{iM}(t)$; $\tilde{\sigma}_{iM}(t) \equiv \sum_{j=1}^d \mathrm{sign}[\rho_{ij}(t)]\sqrt{1-(\rho_{ij}(t))^2}\sigma_i(t)\sigma_j(t)\phi_j^*(t)$; $\tilde{\sigma}_{MM}(t) \equiv \sum_{i=1}^d \phi_i^*(t)\tilde{\sigma}_{iM}(t)$; $\beta^\theta \equiv \frac{\sigma_{iM}(t)+\tilde{\sigma}_{iM}(t)\Phi^2}{\sigma_{MM}(t)(t)+\tilde{\sigma}_{MM}(t)\Phi^2}$.

Proof. See Appendix B.

Theorem 1 suggests that market sentiment will distort the attitude of the agent towards individual securities, as well as the overall financial market, in terms of risk and reward. As sentiment deviating from neutral, expectile CAPM can still be represented as a single factor model; however, the risk premiums are adjusted by the market sentiment; the systematic risk being priced for the market is the weighted sum of the comonotonic risk $\sigma_{MM}(t)$ and the non-comonotonic risk $\tilde{\sigma}_{MM}(t)$, where the weights are completely determined by the market sentiment. With the same set of weights, the systematic risk for any security i is the weighted sum of the comonotonic risk $\sigma_{iM}(t)$ and the non-comonotonic risk $\tilde{\sigma}_{iM}(t)$. Subsection 4.2 will further demonstrate how the expectile CAPM contributes to a potential explanation of the momentum and long-term reversals.

Remark 1. Eq (12) indicates that the agent's sentiment will change the optimal portfolio process $\varpi^*(t)$ directly and, correspondingly, change the total wealth $X(\cdot)$ of the next period through portfolio rebalancing over time. Although the agent's sentiment θ does not enter into the formulation of optimal consumption process $c^*(t)$ explicitly, Eq (13) suggests that $c^*(t)$ is usually a function of total wealth $X(t)$ at that time, through which sentiment may affect the optimal consumption process indirectly.

Remark 2. Eq (12) and Eq (14) suggest that the adjustments to both the portfolio process and the risk premium caused by sentiment are proportional to the square root of the rebalancing frequency, which also influences the risks (systematic, comonotonic, non-comonotonic) indirectly through changing the composition of the optimal market portfolio. Eq (14) implies that it is rational for the agent to trade more (less) often if market sentiment is optimistic (pessimistic). This is an extension of the Merton (1973) paper, which claimed that the equilibrium is a function of the trading intervals chosen owing to the market structural change, e.g., a term structure will illustrate the point. In our context, a static market structure (e.g., a flat rate) with a non-neutral constant market sentiment could also require an optimal holding period before the agent revises her portfolio.

Remark 3. The variance-covariance matrix describes how data is spread across the feature space. The largest eigenvalue is the magnitude of the vector that points into the direction with the largest spread. The second largest eigenvalue is the magnitude of the vector orthogonal to the largest eigenvector, and points in the direction of the second largest spread, and so on. The intuitive geometric interpretation of the required condition in Proposition 1, that θ ensures V_θ p.s.d., is that although the sentiment will reshape agent's view of the data distribution, she holds the basic reason for the non-negative magnitudes of the reshaped data spreads.

Remark 4. See Appendix A, the proof of Proposition 1 indicates that the *sentiment portfolio optimization* in the original market (denoted as $\mathscr{M}$) is equivalent to the classical *portfolio optimization* in an auxiliary market (denoted as $\mathscr{M}_\theta$), where the returns on primary assets follow geometric Brownian motion with drift rate vector, $\tilde{b} = b + \Phi\sqrt{n}\sigma$, and variance-covariance matrix V_θ. Such equivalence provides a way of translating a *sentiment portfolio optimization* into a *classical portfolio optimization.* Then, the existing asset pricing knowledge can be employed directly.

Definition 4. The classical portfolio optimization problem in auxiliary market $\mathscr{M}_\theta$ is

$$\mathbb{J}(x_t, t) \triangleq \underset{(\varpi,c)\in\mathscr{A}_0'(x_t,t,T)}{\mathrm{esssup}} \tilde{\mathbb{E}}\left[\int_t^T U_1[c(s), s]ds + U_2(X^{x_t,\varpi,c}(T))|\mathscr{F}_t\right] \tag{15}$$

where $\tilde{\mathbb{E}}(\cdot)$ represents the expectation under the probability measure in the auxiliary market $\mathscr{M}_\theta$, where the primary assets S_i^θ, $i = 1, \dots, d$, follow the dynamics of

$$dS_i^\theta(t) = S_i^\theta(t)\left[\tilde{b}_i dt + \sum_{j=1}^d \eta_{ij} d\widehat{W}_i(t)\right], \qquad S_i^\theta(0) = s_i, \quad i = 1, 2, \dots, d \tag{16}$$

where $d \times d$ matrix η is the lower triangular Cholesky factorization of V_θ, and $d\widehat{W}(t) = \eta^{-1}\mathrm{diag}(\sigma)dW(t)$, and $\mathrm{diag}(\sigma)$ is the diagonal matrix with the elements of vector σ as its diagonal entries.

So far, we assume there is only one representative agent. First, does there exist a representative investor if all investors have the same sentiment? Second, in practice, individual heterogeneity may drive differing sentiment, then how to aggregate them? As the discussion does not change the results henceforth, we discuss the aggregation in Appendix C.

4.3. A new pricing kernel

Araujo et al. (2012) developed an Arrow–Debreu ambiguous state price of single-period securities markets with finitely many states. In this subsection, we create an extended asset pricing framework by identifying a new pricing kernel incorporating agent's sentiment, assuming continuous time and uncountable many states.

Theorem 2. *Given a market sentiment θ such that the $d \times d$ matrix V_θ is p.s.d., the fair price of a contingent claim with terminal payoff $B(T)$ is uniquely determined by*:

$$p \equiv p(x_t, t, T) = \frac{\tilde{\mathbb{E}}\left[U_2'\left(X^{x_t,\varpi^*,c^*}(T)\right)B(T)|\mathscr{F}_t\right]}{U_1'[t, c^*(t)]}, \tag{17}$$

where $\tilde{\mathbb{E}}[\cdot]$ represents the expectation under the probability measure in auxiliary market $\mathscr{M}_\theta$.

Proof Directly from Remark 4 and Theorem 7.2 in Karatzas and Kou (1996).

Theorem 3. Given a market sentiment θ such that the $d \times d$ matrix V_θ is p.s.d., the fair price of a contingent claim with terminal payoff $B(T)$ is uniquely determined by:

$$p \equiv p(x_t, t, T) = \tilde{\mathbb{E}}\left[\frac{H(T)B(T)}{H(t)}\middle|\mathscr{F}_t\right], \tag{18}$$

where $\tilde{\mathbb{E}}[\cdot|\mathscr{F}_t]$ represents the conditional expectation under the probability measure in the auxiliary market $\mathscr{M}_\theta$, where in auxiliary market $\mathscr{M}_\theta$ is $H(t) = \exp\{-rt\}\exp\left\{-\int_0^t \vartheta_\theta \cdot d\widehat{W}(s) - \frac{1}{2}\int_0^t \vartheta_\theta^2 ds\right\}$, for $0 \le s \le t$, where $\vartheta_\theta = \eta^{-1}[\tilde{b} - r1]$.

ProofDirectly from Remark 4 and Theorem 7.4 in Karatzas and Kou (1996).

Theorem 3 suggests that market sentiment affects both the agent's prior of the payoff being priced and the pricing kernel: the one incorporates and summarizes the prior of all primary assets.

If we make a further assumption that the agent consumes all the terminal wealth at time T, Theorem 2 and Theorem 3 indicate a transformation between the new asset pricing kernel considering sentiment, and the consumption growth. From there, we will see in Subsection 4.3 how the equity premium puzzle arises.

4.4. Implementations for empirical literature

Stambaugh et al. (2012) documented the role of sentiment empirically in a broad set of anomalies in cross-sectional stock returns. In this section, we apply expectile asset pricing framework to examine, theoretically, its implication for the pricing kernel puzzle, the relationship between momentum-induced profits and sentiment, and propose a model illustrating how sentiment inflates the equity premium puzzle. The purpose of this study is not resolving anomalies, as being exhaustive in examining all of the potential explanations and claiming expectile framework is empirically reconcilable with the anomalies is beyond the scope of this theoretical paper. We shed light on a novel channel to invoke further creative empirical research.

4.5. Pricing kernel puzzle

The pricing kernel summarizes all relevant asset pricing information. It is a change of measure (see Eq (18)), as well as the marginal rate of substitution (see Eq (17)) i.e., the rate at which the investor is willing to substitute consumption at a future time for consumption at the current moment. It is well-known that the classical CAPM can be reformulated as a linear relationship between the pricing kernel m and the market gross return R^W (see Cochrane, 2010, p 133),

$$m = A - B \times R^W, \tag{19}$$

with

$$m = \begin{cases} A = \dfrac{1}{R_t^f} + b\mathbb{E}_t\left(R_{t+1}^W\right) \\ B = \dfrac{\mathbb{E}_t\left(R_{t+1}^W\right) - R_t^f}{R_t^f VAR_t\left(R_{t+1}^W\right)} \end{cases} \tag{20}$$

where R_t^f is the gross risk-free rate and $VAR_t(R_{t+1}^W)$ is the conditional variance of R_{t+1}^W.

In a normal market, obviously B is positive, hence, the pricing kernel declines as market return rises. The pricing kernel puzzle refers to the observation that the pricing kernel might increase in some range of the market returns (Ait-Sahalia and Lo, 2000; Jackwerth, 2000).

Most extant explanations for this puzzle are based on missing risk factors (e.g. Chabi-Yo et al., 2008), heterogeneous beliefs (e.g. Shefrin, 2008; Ziegler, 2007; Hens and Reichlin, 2013), statistical and estimation issues (Siddiqi and Amwar, 2018), as well as utilities, such as non-concave utilities (Hens and Reichlin, 2013), rank-dependent expected utilities (Polkovnichenko and Zhao, 2013) and state dependent utilities (Krishna and Sadowski, 2014), and the behavioral channel has not been explored.[5]

We claim that the affine transformation imposed in Remark 4 (from b to $\tilde{b}$) can generate a non-monotonic pricing kernel. In particular, a simple way to incorporate sentiment is to consider the classical portfolio optimization in the auxiliary market. In that case, the B in Eq (20) will be adjusted into

$$B = \frac{\mathbb{E}_t\left(R_{t+1}^W\right) + \Phi\sqrt{n}[\phi^*(t)]^\top \sigma - R_t^f}{R_t^f[\sigma_{MM}(t)(t) + \tilde{\sigma}_{MM}(t)\Phi^2]} \tag{21}$$

[5] Polkovnichenko and Zhao (2013) might be the only exception.

Then, depending on the sign of $[\phi^*(t)]^\top \sigma$, as sentiment deviates from neutral, $\Phi\sqrt{n}[\phi^*(t)]^\top \sigma$ can be negative and consequently make B fall into a negative zone. Hence, the non-monotonicity of the pricing kernel with respect to the market sentiment can be observed in some range of sentiment. Then, owing to the strong correlation between sentiment levels and contemporaneous market returns, as documented in Brown and Cliff (2004), the non-monotonicity can also be observed in some range of the market returns. That could be a possible explanation for the pricing kernel puzzle.

4.6. Momentum and long-term reversal

Cross-sectional momentum, the only anomaly unexplained by the three-factor model (Fama and French, 1996), was first documented by Jegadeesh and Titman (1993). Winners (i.e., stocks with strong past performance based on 6-month lagged returns and held for 6 months) continue to outperform losers over the next period on a 3- to 12- month horizon, with betas (also known as the relative systematic risk) for the winners being even lower than those for the losers, and the effect dissipates after 12 months. De Bondt and Thaler (1985, 1987) found long-term reversals in cross-sectional returns over 2- to 5-year horizons.

Many risk-based explanations for momentums have been proposed, either theoretically or empirically (Johnson, 2002; Sagi and Seasholes, 2007). The existing empirical literature of momentum is far from conclusive. For example, momentum strategies are highly volatile and experience infrequent but severe losses in panic states (Daniel and Moskowitz, 2016), which challenges existing rational or behavioral explanations. In this subsection, we explain the momentum using the extended asset pricing framework developed in Section 3.

The expectile CAPM identifies three sources of misspecification inherent in the classical CAPM: 1) unadjusted risk premium for security, 2) unadjusted market risk premium, and 3) degenerated composition of systematic risk. We propose the following two alternative channels to explain the remaining momentum in profits, after controlling the first misspecification by creating winner and loser portfolios with the same volatility.

Hypothesis 1. (*Market-risk-premium approach*) After adjustment for market sentiment, the systematic risks for the winner and loser portfolios change in magnitude, but the ranking remains the same. However, the adjusted market risk premium (by a pessimistic view) becomes negative, and that makes the portfolio with greater systematic risk become a loser.

Hypothesis 2. (*Systematic risk approach*) After adjustment for market sentiment, the market risk premium is still positive. However, the non-comonotonic risk becomes a large proportion of the systematic risk. Although the winner portfolio has a low comonotonic risk (usually understood as the systematic risk in the classical CAPM), it has a high non-comonotonic risk, which dominates in determining the systematic risk, while market sentiment is incorporated. That substantially changes the rankings of the profitability of the stocks.

Hypotheses 1 and 2 provide alternative reasoning for the fact that the winners are often associated with betas even lower than those of the losers. In addition, the formation and updating of market sentiment can be an overcorrecting procedure; a pessimistic or optimistic view will be identified within a short period and fully corrected over a longer term: that is consistent with the striking term structure, found in Han and Li (2017) using China market data, that local sentiment shifts from a short-term momentum predictor to a contrarian predictor in the long run. Those facts, together, resolve momentum and long-term reversal anomalies.

Many studies report on using idiosyncratic risk to explain the persistence of momentum and reversal, e.g., Arena et al. (2008), who found a positive correlation between momentum return and idiosyncratic volatility, and McLean (2010) reported that the prevalence of reversal is found only in high idiosyncratic risk stocks, and claimed the reason that idiosyncratic risk limits arbitrage in reversal mispricing. We argue that idiosyncratic risk is defined as risk that can be mitigated through diversification; hence, by Definition, taking idiosyncratic risks should not be rewarded by a higher return. Expectile CAPM suggests that the non-comonotonic risk, which is rewarded as it forms part of the systematic risk when market sentiment deviates from neutral, might have been misunderstood as idiosyncratic.

To illustrate the above point, for convenience, we consider the contribution from security j of security i's systematic risk by projecting the total risk of security i on security j (rather than on the market portfolio), see Fig. 1 below. Summing the projections over j, the systematic risk is obtained by Definition.

Fig. 1(a) indicates that when market sentiment is neutral, the contributions of systematic risk and comonotonic risk coincide as σ_{ij}; Fig. 1(b) shows when market sentiment is optimistic or pessimistic, the agent's view will cast with a squint angle, and project on security j, resulting in a systematic risk contribution $\sigma_{ij}+\tilde{\sigma}_{ij}\Phi^2$, which is a mixture of comonotonic σ_{ij} and non-comonotonic $\tilde{\sigma}_{ij}$. In that case, the idiosyncratic risk contribution is formulated as: $\tilde{\sigma}_{ij}-\sigma_{ij}\Phi^2$, and is still independent of the systematic risk contribution. In sum, when market sentiment is non-neutral, non-comonotonic risk will be priced. However, the idiosyncratic risk should never be priced.

4.7. Equity premium puzzle

Mehra and Prescott (1985) first documented the equity premium puzzle as the fact that, to rationalize Post World War II US capital market and consumption data, the mean-variance based portfolio theory requires an implausibly high degree of risk aversion. We claim that the anomaly can be inflated by pessimistic market sentiment.

Proposition 2. Given market sentiment θ such that the $d\times d$ matrix V_θ is p.s.d., the following inequality holds:

$$b_M+\Phi\sqrt{n}[\phi^*(t)]^\top\sigma-r<\gamma\sigma_c\sqrt{\sigma_{MM}(t)+\tilde{\sigma}_{MM}(t)}, \tag{22}$$

where $\gamma\equiv-\frac{c^*(t)U_1^{''}(t,c^*(t))}{U_1^{'}(t,c^*(t))}$ is the Arrow-Pratt measure of relative risk-aversion.

Proof. See Appendix D.

It is easy to see that a further assumption, i.e., neutral market sentiment ($\theta=50\%$), will make Eq (22) degenerate into the classical equity premium puzzle, formulated as below:

$$b_M-r<\gamma\sigma_c\sigma_M, \tag{23}$$

where $\sigma_M(t)$ is the volatility of market portfolio with $\sigma_M(t)\equiv\sqrt{\sigma_{MM}}$.

Proposition 2 suggests that market sentiment could be the reason the equity premium puzzle is inflated. Fear of the unknown, increases the systematic risk by introducing non-comonotonic risk additional to the comonotonic risk. Such pessimistic beliefs also dissipate the market risk premium and, hence, to keep the same level of Sharpe ratio, a much greater expected return is required. That explains why the observed returns on stocks in the second half of the 20th century are much higher than the returns on government bonds.

The risk-free rate puzzle of Weil (1989), presented as the inverse of the equity premium puzzle, questions why government bond returns are lower than equity returns to a level beyond what a plausible risk aversion could explain. We claim that, assuming instantaneous risk-free rate r is non-stochastic, then by the Definition of $H(t)$ and by the fact that the gross risk-free rate: $R_f=1/\mathbb{E}^\theta(H(t))=1/\mathbb{E}(H(t))$, it follows that R_f is not affected by market sentiment. According to Eq (12), pessimistic sentiment will decrease all the elements in portfolio process vector, $\pi^*(t)$ and, thus, the proportion of the wealth assigned to the risk-free asset will be increased. That explains why investors seeking out high returns invest heavily in government bonds, rather than in equities. This is consistent with the empirical finding in Da et al. (2015), that sentiment, proxied by the Financial and Economic Attitudes Revealed by Search (FEARS) index, predicts mutual fund flows out of equity funds and into bond funds. If the investors did invest in fewer equities, returns from the equities would rise, causing the returns for government bonds to fall, and making the equity premium even greater.

5. Conclusion

This research is driven by the contradiction between the noise-trader sentiment risk Hypothesis and the high predictability of market sentiment owing to the social media nowadays, as well as by the continuing debates on the pricing kernel puzzle, momentum, long-term reversal, and the equity premium puzzle. We contribute to a potential explanation for these debates from the new perspective of sentiment emanating from the use of psychological insight by rational, sophisticated investors. In particular, we define *sentiment portfolio optimization*, and state the equivalence between *sentiment portfolio optimization* in the original market and *classical portfolio optimization* in the auxiliary market. This equivalence leads to the expectile CAPM and two kernel pricing formulas. From there, we propose explanations for the anomalies and puzzles stated above.

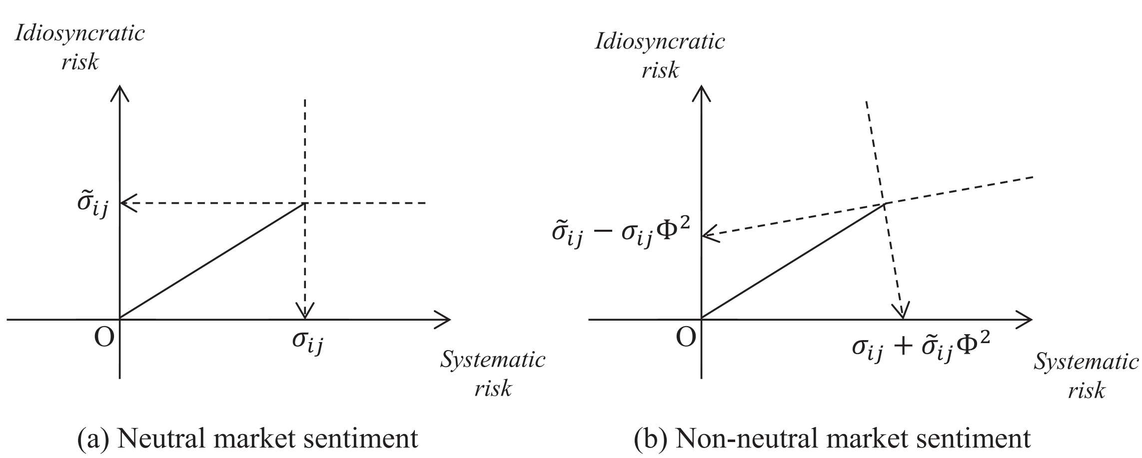

Figure 1. Systematic risk and idiosyncratic risk decomposition. (a) Neutral market sentiment ($\theta=50\%$); (b) Non-neutral market sentiment, either optimistic ($\theta>50\%$) or pessimistic ($\theta<50\%$).

The essence of the impacts of market sentiment lies in its ability to change the monetary value of information, optimal wealth allocation and consumption, risk decomposition, portfolio rebalancing frequency, etc. All in all, it changes both the quantity of risk and the market price of the risk. The main arguments include: 1) pricing theory extends the classical *Law of One Price* in such a way that it is profitable to repackage assets and sell the portfolio (or to buy a portfolio and sell its content) when market pessimism (optimism) prevails. The instantaneous profits earned by doing so, reward the party having information superiority for her information release to the market. 2) Optimistic (pessimistic) sentiment encourages agents to take a greater (lesser) proportion of risky assets and, hence, a lesser (greater) proportion of risk-free assets. Such changes associated with trading activity result in a different total wealth, which, in turn, affects the agent's consumption behavior. 3) When the market is sentiment-neutral, the systematic risk coincides with the comonotonic risk, and the idiosyncratic risk coincides with the non-comonotonic risk. Non-comonotonic (comonotonic) risk enters into the systematic (idiosyncratic) risk formulation in addition to the comonotonic (non-comonotonic) risk, and increases in proportion[6] as the market sentiment deviates further from neutral to either pessimistic or optimistic. 4) Idiosyncratic risk is always independent of systematic risk, hence, it is not priced, regardless of whether or not market sentiment is neutral. 5) The risk premium increases as market sentiment becomes more optimistic, and decreases as it becomes more pessimistic, and the change is proportional to the square root of the rebalancing frequency. 6) When pricing an asset, the market sentiment affects both the agent's prior of its payoff and the pricing kernel, which incorporates and summarizes the prior of all primary assets.

The implementation on explaining the pricing kernel puzzle demonstrates the empirical relevance of the proposed pricing kernel. The affine transformation, bridging the original market and the auxiliary market augmented for adapting sentiment, can generate a non-monotonic pricing kernel.

As a result of conducting this study, we propose two conditions causing the momentum to persist: 1) a negative market risk premium caused by very pessimistic sentiment; and 2) a non-comonotonic-risk-dominated systematic risk that suggests a different ranking of the profitability of stocks. However, if these two conditions are concurrently satisfied, the momentum can be mitigated.

This study suggests a possible answer to the equity premium puzzle, i.e., why the market excess return is high to a level beyond what a plausible risk aversion could explain. After World War II, on the one hand, the 'fear-of-the-unknown' atmosphere made it impossible for non-comonotonic risk to be fully diversified, and that prompted investors to ask for a higher return, commensurate with the greater systematic risk; on the other hand, the returns on assets were often discounted, because investors held a pessimistic view of future returns as forecasted using historical data, especially for highly volatile stocks. Hence, the market-required returns for risky assets were even greater. Such mispricing cannot be arbitraged by trading activities seeking high profits. On the contrary, doing that would exaggerate the puzzle, unless the pessimistic sentiment were to be corrected over a period of time.

The proposed theoretical results can be tested empirically for future research. First, given market sentiment as a constant, extending quantile regression by revising the minimization objective from an asymmetrically weighted mean absolute error to an asymmetrically weighted mean squared error, enables the development of an expectile regression for testing the expectile CAPM empirically. By analogy, extending the Bayesian approach applicable to the quantile regression to make it work for the expectile regression, enables us to relax the constant market sentiment assumption, and to use the developed Bayesian approach to estimate the unknown market unknown parameter, such as market sentiment and sentiment-adjusted beta, or to filter them out as latent state variables. That, to some extent, alleviates CAPM's shortcoming that it can be tested only with historical data, which against the fact that the risks and risk premia in CAPM are all ex ante. We can further conduct a GMM estimation with the expectile CAPM and pricing anomalies, such as the pricing kernel puzzle, equity premium puzzle, etc., as moment conditions, and determine the level of market sentiment required to mitigate those anomalies.

The main policy recommendations are stated as follows. Educating "naïve" investors about their behavioral biases may not enhance the efficiency of equity market prices effectively, because sentiment can also be brought about by sophisticated investors' use of psychological insight. Policy makers should pursue strategies of influencing investors' understanding of the new efficiency structure, and encouraging them to invest under the constraints of existing sentiment.

Appendix A. Proof of Proposition 1.

According to the dynamic programming principle,

$$\mathbb{J}(x_t,t)=\underset{\left(\varpi,c\right)\in\mathscr{A}_0^{\cdot}(x_t,t,\ t+dt)]}{\text{esssup}}\mathbb{E}^{\theta}[\mathbb{J}(X(t+dt),t+dt))|\mathscr{F}_t] \tag{A.1}$$

We apply Taylor expansion to $\mathbb{J}(X(t+dt),t+dt))$ in Eq (A.1) to reduce the dimension and derive HJB

$$0=\underset{(\varpi,c)\in\mathscr{A}_0^{\cdot}(x_t,t,\ t+dt)}{\text{esssup}}U_1[c(t),\quad t]dt+\mathbb{J}(X(t),t)dt+\mathbb{J}_X(X(t),t)\mathbb{E}^{\theta}\left[dX|\mathscr{F}_t]+\frac{1}{2}\mathbb{J}_{XX}(X(t),t)\mathbb{E}^{\theta}[dXdX|\mathscr{F}_t\right]+O\left((dt)^2\right) \tag{A.2}$$

We add the expectation operator on Eq (7), and get

$$\mathbb{E}^{\theta}\left[dX(t)|\mathscr{F}_t\right]=\left\{\sum_{i=1}^{d}\varpi^i(t)X(t)[b_i+\sigma_i\mathbb{E}^{\theta}[dW_i(t)|\mathscr{F}_t]\right]+\left[1-\sum_{i=1}^{d}\varpi^i(t)\right]X(t)r-c(t)\Big\}dt \tag{A.3}$$

and

[6] It can be a negative proportion and, in that case, "greater" refers to the magnitude of the proportion.

$$\mathbb{E}^\theta\left[dX(t)dX(t)|\mathscr{F}_t\right] = [X(t)]^2\left(\sum_{i=1}^{d}\sum_{j=1}^{d}\left[\varpi^i(t)\varpi^j(t)\sigma_i\sigma_j\right]\mathbb{E}^\theta\left[dW_i(t)dW_j(t)\middle|\mathscr{F}_t\right]\right)dt \tag{A.4}$$

Taking the derivative of which *w.r.t.* $c(t)$ and $\varpi(t)$, we get the following first-order-conditions (FOCs):

$$U_{1,c}[c(t),t] - \mathbb{J}_X(X(t),t) = 0 \tag{A.5}$$

and for $i = 1,2,\ldots,d$,

$$\mathbb{J}_X(X(t),t)\left\{b_i + \frac{\Phi}{\sqrt{dt}}\sigma_i - r\right\} + \mathbb{J}_{X,X}(X(t),t)X(t)\left\{\sum_{j=1}^{d}\left[\varpi^j(t)\rho_{ij}\sigma_i\sigma_j\right]\Psi + \sum_{j=1}^{d}\left[\varpi^j(t)\mathrm{sign}\left[\rho_{ij}\right]\sqrt{1-\rho_{ij}^2}\,\sigma_i\sigma_j\right]\Phi^2\right\} = 0 \tag{A.6}$$

where $\Phi \equiv \mathbb{E}^\theta[\varepsilon]$, and ε is a $\mathscr{N}(0,1)$ random variable, and $\Psi \equiv \mathbb{E}^\theta[\varepsilon^2]$. Numerical approximation of Ψ indicates $\Psi \equiv 1$, for any value of $\theta \in [0,1]\Psi \equiv \mathbb{E}^\theta[\varepsilon^2] = \mathbb{E}[\pi^\theta(\varepsilon)\varepsilon^2]\mathbb{E}[\pi^\theta(z)z]z = \varepsilon^2$.

Denoting $1/\sqrt{dt}$ as rebalancing frequency n, Eq (A.6) implies that *sentiment portfolio optimization* in the original market (denoted as $\mathscr{M}$) is equivalent to the classical *portfolio optimization* in an auxiliary market (denoted as $\mathscr{M}_\theta$), where the returns of primary assets follow geometric Brownian motion with drift rate vector, $\tilde{b} = b + \Phi\sqrt{n}\sigma$, and variance-covariance matrix V_θ.

Rewriting Eq (A.6) as a more compact expression, we then get the optimized portfolio process,

$$\varpi^*(t) = -\frac{\mathbb{J}_X(X(t),t)}{X(t)\mathbb{J}_{X,X}(X(t),t)}V_\theta^{-1}\left\{b + \Phi\sqrt{n}\,\sigma - r1\right\} \tag{A.7}$$

where $V_\theta = (\rho_{ij}\sigma_i\sigma_j + \mathrm{sign}(\rho_{ij})\sqrt{1-\rho_{ij}^2}\,\sigma_i\sigma_j\Phi^2)$, $i,j = 1,2,\ldots,d$.

This completes the proof of Proposition 1.

Appendix B. Proof of Theorem 1.

We denote investor k's optimized portfolio process, wealth process and indirect utility as $\varpi_k^*(t)$, $X_k(t)$ and $\mathbb{J}^k(X(t),t)$, and sum K homogeneous investors' portfolio weights, then we get the aggregated market portfolio process $\pi_M^*(t)$,

$$\varpi^*(t) = \frac{\sum_{k=1}^{K}\varpi_k^*(t)X_k(t)}{\sum_{k=1}^{K}X_k(t)} = \frac{A}{\mathbb{X}}V_\theta^{-1}\left\{b + \Phi\sqrt{n}\,\sigma - r1\right\} \tag{A.8}$$

where

$$A = \sum_{k=1}^{K}\left(-\frac{\mathbb{J}_X^k(X(t),t)}{\mathbb{J}_{X,X}^k(X(t),t)}\right);\ \mathbb{X} = \sum_{k=1}^{K}\ X_k(t) \tag{A.9}$$

Then, the expectile excess return vector satisfies the following equation,

$$\left\{b + \Phi\sqrt{n}\,\sigma - r1\right\} = V_\theta\varpi^*(t)\frac{\mathbb{X}}{A} \tag{A.10}$$

Then, for any security, i, we have the expectile CAPM as follows,

$$b_i + \Phi\sqrt{n}\sigma_i - r = \beta^\theta\left(b_M + \Phi\sqrt{n}\,[\phi^*(t)]^\top\sigma - r\right) \tag{A.11}$$

where $\beta^\theta \equiv \frac{\sigma_{iM}(t)+\tilde{\sigma}_{iM}(t)\Phi^2}{\sigma_{MM}(t)+\tilde{\sigma}_{MM}(t)\Phi^2}$.

This completes the proof of Theorem 1.

Appendix C. Heterogeneity and aggregation

In practice, individual heterogeneity may drive differing sentiment. Hong and Stein (2007) stressed that heterogeneous sentiment can extract different values from new information available to all investors simultaneously. Edelen et al. (2010) stated the difference between the sentiments of retail and institutional investors. Harris and Raviv (1993) and Karpoff (1986) documented a positive relationship between the sentiment disagreement and trading volume. Atmaz and Basak (2018) developed a dynamic belief dispersion model, with a continuum of investors differing in beliefs. Banerjee and Kremer (2010) considered learning patterns of trade from the disagreement in sentiment.

In this subsection, we assume a framework of CARA, which ensure that there exist a representative investor if all investors have the same sentiment θ. We then assume heterogeneous investors, and aggregate individual heterogeneity to construct a representative agent, whose sentiment (risk aversion, initial wealth) is the composite of the sentiments (risk aversions, initial wealth) of individual investors.

Assume K investors are holding the same prior, but with different sentiment θ_k, utilities U_1^k and U_2^k, initial wealth x_t^k at time t, and price-of-risk vector $\varphi_k(t)$, where

$$\varphi_k = V_{\theta_k}^{-1}\left[b + \Phi_k\sqrt{n}\,\sigma - r1\right] \tag{A.12}$$

We denote investor k's optimized portfolio process, wealth process and indirect utility as $\pi_k^*(t)$, $X_k(t)$ and $\mathbb{J}^k(X(t),t)$, where $k = 1,2,\ldots K$. We further assume U_1^k is constant absolute risk aversion (CARA) with an Arrow-Pratt measure of absolute risk-aversion, A_k. Then, the first order condition (FOC) of the *sentiment portfolio optimization* implies that

$$-\frac{\mathbb{J}_X^k(X(t),t)}{\mathbb{J}_{X,X}^k(X(t),t)} = -\frac{U_{1,c}^k[c^k(t),t]}{U_{1,c,c}^k[c^k(t),\ t]} = A_k \tag{A.13}$$

Substituting Eq (A.13) into $\varpi_k^*(t)$, and then summing it over all individuals, we get the aggregated market portfolio process as:

$$\overline{\varpi^*}(t) = \frac{\sum_{k=1}^K \varpi_k^*(t)X_k(t)}{\sum_{k=1}^K X_k(t)} = \overline{R}(t)\overline{\varphi} \tag{A.14}$$

where $\overline{R}(t)$ is the reciprocal of the aggregated Arrow-Pratt measure of relative risk-aversion, and $\overline{\varphi}$ is the aggregated market price of risk,

$$\overline{R}(t) = \frac{\sum_{k=1}^K A_k(t)}{\sum_{k=1}^K X_k(t)};\ \overline{\varphi} = \sum_{k=1}^K\left\{\frac{\varphi_k A_k}{\sum_{k=1}^K A_k}\right\} \tag{A.15}$$

Definition 4. The *representative agent's sentiment* θ is defined as the least bullish or bearish projection of the aggregated sentiment,

$$\theta = \underset{\widehat{\psi}\in\widehat{\Theta}}{\operatorname{argmin}}|\widehat{\psi} - 50\%|, \tag{A.16}$$

where $\Theta \triangleq \{\psi \in [0,1];\ V_\psi \ is\ p.s.d.\}$, $\widehat{\Theta} \triangleq \{\widehat{\psi};\widehat{\psi} = \underset{\psi\in\Theta}{argmin}\overline{\varphi} - \varphi^{\psi 2}\}$, $||\cdot||$ is Euclidian.

Lemma 1. There exists a unique representative agent's sentiment.

Proof. By Definition $V_\psi = V_1 + \Phi^2 V_2$, where $V_1 \equiv (\rho_{ij}\sigma_i\sigma_j)$ and $V_2 \equiv (sign(\rho_{ij})\sqrt{1-(\rho_{ij})^2}\,\sigma_i\sigma_j)$ are Hermitian matrices, then according to Garding (1959), the least eigenvalue of V_ψ is a concave function. Then, Θ is a nonempty closed convex set (more specifically, a closed interval). Because $\overline{\varphi}-\varphi^{\widehat{\psi}}$ is a continuous function of $\widehat{\psi}$, then it attains its minimum at some point(s) contained in the interval, namely the set $\widehat{\Theta}$ is nonempty. Then, it is trivial that Eq (A.16) has a unique solution.

This completes the proof of Lemma 1.

Appendix D. Proof of Proposition 2.

For any contingent claim $B(\cdot)$, we denote the terminal payoff $B(T)$ as p_T, then we can rewrite Eq (18) as

$$p_t H(t) = \tilde{\mathbb{E}}[p_T H(T)|\mathscr{F}_t] \tag{A.17}$$

Hence, process $p_t H(t)$ is a martingale in auxiliary market $\mathscr{M}_\theta$, then we have

$$0 = \tilde{\mathbb{E}}\left[\frac{d(p_t H(t))}{p_t H(t)}|\mathscr{F}_t\right] = \tilde{\mathbb{E}}\left[\frac{dH(t)}{H(t)}\middle|\mathscr{F}_t\right] + \tilde{\mathbb{E}}\left[\frac{dp_t}{p_t}|\mathscr{F}_t\right] + \tilde{\mathbb{E}}\left[\frac{dH(t)}{H(t)}\frac{dp_t}{p_t}\middle|\mathscr{F}_t\right] \tag{A.18}$$

By the Definition of $H(t)$ in Theorem 3,

$$-\frac{dH(t)}{H(t)} = rdt + \vartheta_\theta d\widehat{W}(t) \tag{A.19}$$

In auxiliary market $\mathscr{M}_\theta$, market portfolio return is

$$\sum_{i=1}^d \frac{\phi_i(t)dS_i(t)}{S_i(t)} = \sum_{i=1}^d[\phi_i(t)\tilde{b}_i(t)dt] + \sum_{j=1}^d\sum_{i=1}^d[\phi_i(t)\eta_{ij}(t)]d\widehat{W}_j(t) \tag{A.20}$$

Substitute Eq (A.19) and Eq (A.20) into Eq (A.18), and thank Cauchy inequality, we have

$$\tilde{\mathbb{E}}\left[\sum_{i=1}^d \frac{\phi_i(t)dS_i(t)}{S_i(t)}|\mathscr{F}_t\right] - rdt = \tilde{\mathbb{E}}\left[-\frac{dH(t)}{H(t)}\sum_{i=1}^d\frac{\phi_i(t)dS_i(t)}{S_i(t)}\middle|\mathscr{F}_t\right] < dt\vartheta_\theta(t)\sqrt{\sum_{j=1}^d\left(\sum_{i=1}^d[\phi_i(t)\eta_{ij}]\right)^2} = dt\vartheta_\theta(t)\sqrt{\sigma_{MM}(t) + \tilde{\sigma}_{MM}(t)} \tag{A.21}$$

According to Cvitanić and Karatzas (1992), by construction, the dynamic of $H(t)$ equals $\mathbb{J}_x(x_t,t)$, then we have,

$$d\mathbb{J}_x(x_t,t) = -r(s)\mathbb{J}_x(x_t,t)dt - \vartheta_\theta(s)\mathbb{J}_x(x_t,t)d\widehat{W}(s) \tag{A.22}$$

Apply Itô's Lemma on Eq (13), we have

$$d\mathbb{J}_x(x_t,t) = dU_1^{'}(t,\ c^*(t)) = U_1^{''}(t,\ c^*(t))dc^*(t) + \frac{1}{2}U_1^{'''}(t,\ c^*(t))dc^*, c^*{}_t \tag{A.23}$$

then,

$$\frac{dc^*(t)}{c^*(t)} = -r(s)\frac{U_1^{'}(t,\ c^*(t))}{c^*(t)U_1^{''}(t,\ c^*(t))}dt - \frac{U_1^{'''}(t,\ c^*(t))}{2c^*(t)U_1^{''}(t,\ c^*(t))}dc^*, c^*{}_t - \vartheta_\theta(s)\frac{U_1^{'}(t,\ c^*(t))}{c^*(t)U_1^{''}(t,\ c^*(t))}d\widehat{W}(s) \tag{A.24}$$

We denote

$$\sigma_c \equiv \sigma(ln(c^*(t))) = -\frac{U_1^{'}(t,\ c^*(t))}{c^*(t)U_1^{''}(t,\ c^*(t))}\vartheta_\theta, \tag{A.25}$$

and then substitute ϑ_θ into Eq (A.21), we have

$$b_M + \frac{\Phi}{\sqrt{dt}}[\phi_M^*(t)]^\top\sigma - r < -\frac{c^*(t)U_1^{''}(t,\ c^*(t))\sigma_c}{U_1^{'}(t,\ c^*(t))}\sqrt{\sigma_{MM}(t) + \tilde{\sigma}_{MM}(t)} \tag{A.26}$$

This completes the proof of Proposition 2.

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